A COMBINATORIAL FORMULA FOR THE EHRHART $h^*$-VECTOR OF THE HYPERSIMPLEX

DONGHYUN KIM

Abstract. We give a combinatorial formula for the Ehrhart $h^*$-vector of the hypersimplex. In particular, we show that $h_d^*(Δ_{k,n})$ is the number of hypersimplicial decorated ordered set partitions of type $(k,n)$ with winding number $d$, thereby proving a conjecture of Nick Early. We do this by proving a more general conjecture of Nick Early on the Ehrhart $h^*$-vector of a generic cross-section of a hypercube.

1. Introduction

For two integers $0 < k < n$, the $(k,n)$-th hypersimplex is defined to be

$$Δ_{k,n} = \{ (x_1, \cdots, x_n) \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, x_1 + \cdots + x_n = k \}.$$  

It is an $(n-1)$-dimensional polytope inside $\mathbb{R}^n$ whose vertices are $(0,1)$-vectors with exactly $k$ 1's. In particular it is an integral polytope. The hypersimplex can be found in several algebraic and geometric contexts, for example, as a moment polytope for the torus action on the Grassmannian, or as a weight polytope for the fundamental representation of $GL_n$.

For an $n$-dimensional integral polytope $P \subset \mathbb{R}^N$, it’s a standard fact from Ehrhart theory that the map $r \to |rP \cap \mathbb{Z}^N|$ is a polynomial function in $r$ of degree $n$. Now consider the Ehrhart series

$$\sum_{r=0}^{\infty} |rP \cap \mathbb{Z}^N| t^r = \frac{h^*(t)}{(1-t)^{n+1}}.$$ 

Knowing that $r \to |rP \cap \mathbb{Z}^N|$ is a polynomial function in $r$ of degree $n$, one can check that $h^*(t)$ is a polynomial of degree at most $n$. Define $h_d^*$ to be the coefficient of $t^d$ in $h^*(t)$. The vector $(h_0^*, \cdots, h_n^*)$ is called the Ehrhart $h^*$-vector of $P$ and $h^*(t)$ is called the $h^*$-polynomial of $P$. A standard result from Ehrhart theory is that $\sum_{i=0}^{n} h_i^*$ equals the normalized volume of $P$.

For a permutation $w \in S_n$, we say $i \in [n-1]$ is a descent of $w$ if $w(i) > w(i+1)$ and define $des(w)$ to be the number of descents of $w$. The Eulerian number $A_{k,n-1}$ is the number of $w \in S_{n-1}$ with $des(w) = k - 1$. A well-known fact about the hypersimplex $Δ_{k,n}$ is that its normalized volume is $A_{k,n-1}$. So we have

$$\sum_{d=0}^{n-1} h_d^*(Δ_{k,n}) = A_{k,n-1}.$$ 

In general, the entries of the $h^*$-vector of an integral polytope are nonnegative integers (see [6]). It has been an open problem for some time to give a combinatorial interpretation of $h^*(Δ_{k,n})$. In [4], Nan Li gave a combinatorial interpretation of $h_d^*(Δ'_{k,n})$, where $Δ'_{k,n}$ is the hypersimplex with the lowest facet removed, using permutations $w \in S_{n-1}$ and their descents, excedances, and covers. In [2], Nick
Early conjectured a combinatorial interpretation for $h^*_d(\Delta_{k,n})$ using hypersimplicial decorated ordered set partitions of type $(k,n)$.

In [3], Katzman computed the Hilbert series of algebras of Veronese type, which gives a formula for the Ehrhart series of the hypersimplex $\Delta_{k,n}$ as a special case. The formula is

$$
\sum_{i \geq 0} \binom{n}{i} \binom{j}{i} (t - 1)^i \left( \sum_{i \geq 0} \binom{n-j}{i(k-i)} k^{-i} \right) \over (1-t)^n
$$

(1.1)

where the notation $\binom{n}{a}$ means the coefficient of $t^b$ in $(1 + t + \cdots + t^{a-1})^n$. For example, when $a = 2$, it becomes an ordinary binomial coefficient. The numerator of (1.1) is the $h^*$-polynomial of the hypersimplex, thus giving an explicit formula for its $h^*$-vector. However, it doesn’t give a combinatorial or manifestly positive formula for the $h^*$-vector.

In this paper, we prove Nick Early’s conjecture by relating it to (1.1). We now explain the conjecture. A decorated ordered set partition $((L_1)_1, \cdots, (L_m)_m)$ of type $(k,n)$ consists of an ordered partition $(L_1, \cdots, L_m)$ of $\{1,2,\ldots,n\}$ and an $m$-tuple $(l_1, \cdots, l_m) \in \mathbb{Z}^m$ such that $l_1 + \cdots + l_m = k$ and $l_i \geq 1$. We call each $L_i$ a block and we place them on a circle in a clockwise fashion then think of $L_i$ as the clockwise distance between adjacent blocks $L_i$ and $L_{i+1}$ (indices are considered modulo $m$). So the total length of the circle is $l_1 + \cdots + l_m = k$. We usually regard decorated ordered set partitions up to cyclic rotation of blocks (together with corresponding $l$). For example, $\{(1,2,7)_2, \{3,5\}_3, \{4,6\}_1\}$ is same as $\{(3,5)_3, \{4,6\}_1, \{1,2,7\}_2\}$. A decorated ordered set partition is called hypersimplicial if it satisfies $1 \leq l_i \leq |L_i| - 1$ for all $i$. For the motivation and more background on decorated ordered set partitions, see [1].

**Example 1.1.** Consider a decorated ordered set partition $\{(1,2,7)_2, \{3,5\}_3, \{4,6\}_1\}$ of type $(6,7)$ (see Figure 1). This is not hypersimplicial as $3 > |\{3,5\}| - 1$.

By inserting empty spots, we can encode the distance information. For example, the (clockwise) distance between $\{1,2,7\}$ and $\{3,5\}$ is 2 so we insert one empty spot on the circle between those blocks. The distance between $\{3,5\}$ and $\{4,6\}$ is 3 so we insert two empty spots. We obtain the figure on the right as a result. Including empty spots, there will be $k = 6$ spots total.

![Figure 1](image-url)
Given a decorated ordered set partition, we define the \emph{winding vector} and the \emph{winding number}. To define the winding vector, let \( w_i \) be the distance of the path starting from the block containing \( i \) to the block containing \( (i + 1) \mod n \). If \( i \) and \( (i + 1) \mod n \) are in the same block then \( w_i = 0 \). In Figure 1 the winding vector is \( w = (0, 2, 3, 3, 1, 0) \).

The total length of the path is \( (w_1 + \cdots + w_n) \), which should be a multiple of \( k \) as we started from 1 and came back to 1 moving clockwise. If \( (w_1 + \cdots + w_n) = kd \), then we define the winding number to be \( d \). In Figure 1 the winding number is 2.

It is known that hypersimplicial decorated ordered set partitions of type \((k, n)\) are in bijection with \( w \in S_{n-1} \) such that \( \text{des}(w) = k - 1 \) (see [5]).

**Conjecture 1.2** ([2], Conjecture 1). The number of hypersimplicial decorated ordered set partitions of type \((k, n)\) with winding number \( d \) is \( h_d^*(\Delta_{k,n}) \).

Next we want to state a more general version of Conjecture 1.2 for a generic cross section of a hypercube.

**Definition 1.3.** For positive integers \( r, k, n \), the generic cross section of a hypercube is

\[
I_{r,k}^n = \{(x_1, \ldots, x_n) \in [0, r]^n \mid \sum_{i=1}^n x_i = k\}.
\]

When \( r = 1 \), \( I_{1,k}^n \) is the hypersimplex \( \Delta_{k,n} \).

**Definition 1.4.** A decorated ordered set partition \( P = ((L_1)_{l_1}, \ldots, (L_m)_{l_m}) \) is \emph{r-hypersimplicial} if \( 1 \leq l_i \leq r |L_i| - 1 \) for all \( i \).

Note that the notions of hypersimplicial and 1-hypersimplicial are equivalent. The decorated ordered set partition \((\{1,2,7\}_2, \{3,5\}_3, \{4,6\}_1\) in Example 1.1 is not hypersimplicial, but it is \( r \)-hypersimplicial for \( r \geq 2 \).

**Conjecture 1.5** ([2], Conjecture 6). The number of \( r \)-hypersimplicial decorated ordered set partitions of type \((k, n)\) with winding number \( d \) is \( h_d^*(I_{r,k}^n) \).

2. **Proof of Conjecture 1.5**

2.1. **A simplification of Katzman’s formula.** Again using the formula for Hilbert series of algebras of Veronese type (see [3]), the Ehrhart series of \( I_{r,k}^n \) is

\[
(1 - t)^n \sum_{i \geq 0} (-1)^i \binom{n}{i} \left( \sum_{j \geq 0} \binom{i}{j} (t - 1)^j \sum_{t \geq 0} \binom{n-j}{l(k-r)} \right) \left. \right|_{t = r^i}
\]

(2.1)

Now we simplify (2.1) to get a simple description for the \( h^* \)-vector of \( I_{r,k}^n \).

**Lemma 2.1.** For positive integers \( n, m, a \), \( \binom{n}{m}_a = \binom{n-1}{m-a}_a + \binom{n-1}{m-a}_a \).

**Proof.** By a combinatorial argument we have, \( \binom{n}{m}_a = \sum_{k=0}^{a-1} \binom{n-k}{m-k}_a \) and \( \binom{n}{m}_a = \sum_{k=0}^{a-1} \binom{n-k}{m-k}_a \). Subtracting these two gives the lemma. \( \square \)

**Proposition 2.2.** For positive integers \( s \) and \( a \),

\[
\sum_{j \geq 0} \binom{s}{j} (t - 1)^j \sum_{l \geq 0} \binom{n-j}{l} t^l = \sum_{l \geq 0} \binom{n}{la-s}_a t^l.
\]
Proof. We proceed by induction on $s$. For $s = 0$, this is a trivial identity. Let’s assume that the proposition holds for $s = u - 1$. That is,

$$\sum_{j \geq 0} \binom{u-1}{j}(t-1)^j \left(\sum_{l \geq 0} \binom{n-j}{la} t^l\right) = \sum_{l \geq 0} \binom{n}{la-u+1} t^l.$$

Now replacing $(n - 1)$ by $n$ and multiplying by $(t - 1)$ we have

$$\sum_{j \geq 0} \binom{u-1}{j}(t-1)^{j+1} \left(\sum_{l \geq 0} \binom{n-1-j}{la} t^l\right) = \sum_{l \geq 0} \binom{n-1}{la-u+1} t^l(t-1).$$

Replacing $(j - 1)$ by $j$ and simplifying the righthand side gives

$$\sum_{j \geq 0} \binom{u}{j-1}(t-1)^j \left(\sum_{l \geq 0} \binom{n-j}{la} t^l\right) = \sum_{l \geq 0} \binom{n-1}{la-u+1} t^l.$$

Summing (2.2) and (2.3), and using Lemma 2.1 gives

$$\sum_{j \geq 0} \binom{u}{j}(t-1)^j \left(\sum_{l \geq 0} \binom{n-j}{la} t^l\right) = \sum_{l \geq 0} \binom{n}{la-u} t^l.$$

Using Proposition 2.2, (2.1) becomes

$$\sum_{i \geq 0} (-1)^i \binom{n}{i} \sum_{l \geq 0} \binom{n}{(l+k)k} t^l (1-t)^n.$$ 

Thus we have

$$h_d^*(I_{r,k}^n) = \sum_{i \geq 0} (-1)^i \binom{n}{i} \binom{n}{(k-r)k} t^l (1-t)^n.$$

In Section 2.2, we will prove Conjecture 1.5 which contains Conjecture 1.2 as a special case when $r = 1$. Since we have an explicit formula for $h_d^*(I_{r,k}^n)$, our strategy is to count the number of $r$-hypersimplicial decorated ordered set partitions of type $(k, n)$ with winding number $d$ and compare the formulas.

2.2. Enumeration of $r$-hypersimplicial decorated ordered set partitions with a fixed winding number.

We start with an elementary lemma, skipping the proof.

Lemma 2.3. The $\mathbb{Z}/n\mathbb{Z}$ action on $\{1, 2, \cdots, n\}$ by cyclic shift does not change the winding number of decorated ordered set partitions.

For example, decorated ordered set partitions ($\{1, 2, 7\}_2, \{3, 5\}_3, \{4, 6\}_1$) and ($\{2, 3, 1\}_2, \{4, 6\}_3, \{5, 7\}_1$) have same the winding number.

Next we will show that a winding vector determines a decorated ordered set partition. We observed that when the winding number is $d$, then $w_1 + \cdots + w_n = kd$. And $0 \leq w_i \leq k - 1$ since the total length of the circle is $k$ ($w_i = k$ would mean that $i$ and $(i + 1)$ are in a same block but in that case $w_i = 0$). It turns out that these are the only restrictions for winding vectors.
Proposition 2.4. Decorated ordered set partitions of type \((k, n)\) with winding number \(d\) are in bijection with elements of \(\{(w_1, \ldots, w_n) \in \mathbb{Z}^n \mid 0 \leq w_i \leq k - 1, \ w_1 + \cdots + w_n = kd\}\).

Proof. It is enough to construct a decorated ordered set partition of type \((k, n)\) with winding number \(d\) from a winding vector satisfying the above conditions. First, draw \(k\) spots on the circle in clockwise order and put 1 in one spot. Having put \(i\) in some spot, move clockwise \(w_i\) spots and put \(i + 1\) in that spot. After placing all elements, nonempty spots become blocks and the clockwise distance from \(L_i\) and \(L_{i+1}\) is \(l_i\). \(\square\)

Example 2.5. For type \((k, n) = (6, 7)\), we will construct a decorated ordered set partition from the vector \((0, 2, 3, 3, 3, 1, 0)\). See Figure 2. First, draw \(k = 6\) spots and put 1 in one spot (upper-left figure). Then put elements according to the given vector (upper-right figure). \{1, 2, 7\}, \{3, 5\}, and \{4, 6\} will be blocks. There is one empty spot between \{1, 2, 7\} and \{3, 5\} so the distance is 2. The distance between \{3, 5\} and \{4, 6\} is 3 as there are two empty spots. Resulting decorated ordered set partition is \((\{1, 2, 7\}_2, \{3, 5\}_3, \{4, 6\}_1)\) (lower figure). We recovered Example 1.1.

From Proposition 2.4, we know that the number of decorated ordered set partitions of type \((k, n)\) with winding number \(d\) is \(|\{(w_1, \ldots, w_n) \in \mathbb{Z}^n \mid 0 \leq w_i \leq k - 1, \ w_1 + \cdots + w_n = kd\}|\). A simple combinatorial argument shows this number is the same as
the coefficient of $t^k$ in $(1 + \cdots + t^{k-1})^n$, which is $\binom{n}{k}$. So the number of decorated ordered set partitions of type $(k, n)$ with winding number $d$ is $\binom{n}{kd} k^d$.

Recall that we are interested in the number of $r$-hypersimplicial decorated ordered set partitions of type $(k, n)$ with winding number $d$. Throughout this section, when we say decorated ordered set partition, we always assume it is of type $(k, n)$ with winding number $d$.

**Definition 2.6.** For a decorated ordered set partition $P = \{(L_1)_{l_1}, (L_2)_{l_2}, \ldots, (L_m)_{l_m}\}$, a block $L_i$ is $r$-bad if $l_i \geq r|L_i|$. Let $I_r(P) = \{L_i \mid L_i \text{ is } r\text{-bad}\}$.

For example, $I_1((\{1, 2, 7\}, \{3, 5\}, \{4, 6\}_1)) = \{\{3, 5\}\}$. Recall that $r$-hypersimplicial decorated ordered set partitions satisfy $1 \leq l_i \leq r|L_i| - 1$ for all blocks. So a decorated ordered set partition is $r$-hypersimplicial if and only if $I_r(P)$ is empty.

**Definition 2.7.** For a set $T$, define $UP(T)$ to be a set of all (unordered) partitions of $T$. For example, $\{\{1, 2, 4\}, \{3\}, \{5\}\} \in UP(\{1, 2, 3, 4, 5\})$.

**Definition 2.8.** For $T \subseteq \{1, 2, \cdots, n\}$ and $S \in UP(T)$, define $K_r(S) = \{P \mid \text{decorated ordered set partition such that } S \subseteq I_r(P)\}$.

In other words $K_r(S)$ is the set of all decorated ordered set partitions having elements of $S$ as $r$-bad blocks. For example, when $S = \emptyset$, $K_r(\emptyset)$ is a set of all decorated ordered set partitions.

**Definition 2.9.** For $T \subseteq \{1, 2, \cdots, n\}$, let $H_r(T) = \sum_{S \in UP(T)} (-1)^{|S||K_r(S)|}$.

For example, when $T = \{1, 2, 3\}$,

\[
H_r(T) = -|K_r(\{\{1, 2, 3\}\})| + |K_r(\{\{1, 2\}, \{3\}\})| + |K_r(\{\{2, 3\}, \{1\}\})| + |K_r(\{\{1\}, \{2\}, \{3\}\})|.
\]

**Proposition 2.10.** The number of $r$-hypersimplicial decorated ordered set partitions is

\[
\sum_{T \subseteq \{1, 2, \cdots, n\}} H_r(T).
\]

**Proof.** It is enough to compute $\sum_{T \subseteq \{1, 2, \cdots, n\}} (\sum_{S \in UP(T)} (-1)^{|S||K_r(S)|})$, by the definition of $H_r(T)$. If decorated ordered set partition $P$ has empty $I_r(P)$ then it will be counted once when $S = \emptyset$. If $I_r(P)$ is non empty, say $|I_r(P)| = m$. Then $P$ will be counted $\binom{m}{i}$ times with weight $(-1)^i$ as $S$ ranges over all $i$-element subsets of $I_r(P)$. So the total weight is $\sum_{i=0}^{m} (-1)^i \binom{m}{i} = 0$. So the above sum counts $P$ such that $I_r(P)$ is empty, which means $r$-hypersimplicial.

When $S \in UP(\{1, 2, \cdots, n\})$, elements of $K_r(S)$ are decorated ordered set partitions $P = ((L_1)_{l_1}, \cdots, (L_m)_{l_m})$ whose blocks are all $r$-bad, which means $l_i \geq r|L_i|$ for all $i$. Summing inequalities for all $i$ gives $\sum l_i \geq r \sum |L_i|$ that implies $k \geq rn$ which is impossible as $k < n$. Thus $K_r(S)$ is an empty set so $H_r(\{1, 2, \cdots, n\}) = 0$. So we will only consider when $T$ is a proper subset of $\{1, 2, \cdots, n\}$. By Lemma 2.3, $H_r(T)$ is invariant under cyclic shifts of $\{1, 2, \cdots, n\}$. We may assume that $n \notin T$.

**Definition 2.11.** For $T \subseteq \{1, 2, \cdots, n\}$, a $T$-singlet block is a block with only one element $t$ and $t \in T$. A sequence of consecutive $T$-singlet blocks $(L_i, \cdots, L_{i+j})$ in a
decorated ordered set partition $P$ (indices are considered modulo number of blocks in $P$) is $r$-packed if $l_i = \cdots = l_{i+j-1} = r$ and $l_{i+j} \geq r$. An $r$-packed sequence is increasing $r$-packed if elements in $(L_i, \cdots, L_{i+j})$ are in increasing order. Such a sequence is maximal if it is not a subsequence of another increasing $r$-packed sequence.

The increasing $r$-packed condition highly depends on $T$ since it only applies to consecutive $T$-singlet blocks. Note that $T$-singlet blocks in $r$-packed sequence are all $r$-bad. It is the most concentrated arrangement that makes these blocks all $r$-bad. We allow increasing $r$-packed sequence of length 1 by convention.

**Example 2.12.** For $T = \{1, 2, 4, 6\}$ and $r = 2$, Figure 3 is the picture for decorated ordered set partition $(\{1\}^2, \{2\}^2, \{4\}^2, \{5, 8, 9, 10\}^1, \{6\}^2, \{7\}^2, \{11, 12, 13\}^1)$. Maximal increasing $r$-packed sequences here are $(\{1\}, \{2\}, \{4\})$ and $(\{6\})$. Note that $(\{6\}, \{7\})$ is not $r$-packed since $\{7\}$ is not a $T$-singlet block.

**Figure 3.** Reading off $r$-packed sequences for $r = 2$.

**Lemma 2.13.** Let $S = \{M_1, M_2, \cdots, M_j\} \in UP(T)$, where $T = \{t_1 < t_2 < \cdots < t_m\}$ and $n \notin T$. Enumerate the elements of $M_i$ in increasing order, so $M_i = \{t_{i_1} < t_{i_2} < \cdots < t_{i_w}\}$. Then elements of $K_r(S)$ are in bijection with elements of $K_r(\{\{t_{i_1}\}, \{t_{i_2}\}, \cdots, \{t_{i_w}\}\})$ having increasing $r$-packed sequence $\{\{t_{i_1}\}, \{t_{i_2}\}, \cdots, \{t_{i_w}\}\}$ for all $i$.

**Proof.** Given a decorated ordered set partition $P \in K_r(S)$, $P$ has a block $(M_i)_l$ which is $r$-bad. So $l \geq r|M_i| = rw$. Change $(M_i)_l$ to $\{t_{i_1}\}, \{t_{i_2}\}, \cdots, \{t_{i_w}\}_{l-r(w-1)}$. Since $l-r(w-1) \geq r$, the sequence $\{\{t_{i_1}\}, \{t_{i_2}\}, \cdots, \{t_{i_w}\}\}$ will be increasing $r$-packed. This process does not change the winding number and new $T$-singlet blocks are all $r$-bad. Repeating this process for all $i$ we get the desired correspondence. \qed

**Example 2.14.** See Figure 4. The figure on the left is a decorated ordered partition $(\{1, 2, 4\}^6, \{5, 8, 9, 10, 13\}^1, \{6, 7\}^4, \{11, 12\}^1)$. When $T = \{1, 2, 4, 6, 7\}$ and $r = 2$, the figure on the left has $r$-bad blocks $\{1, 2, 4\}$ and $\{6, 7\}$, so belongs to $K_r(\{\{1, 2, 4\}, \{6, 7\}\})$. Under the correspondence stated in Lemma 2.13, this goes to $(\{1\}^2, \{2\}^2, \{4\}^2, \{5, 8, 9, 10, 13\}^1, \{6\}^2, \{7\}^2\{11, 12\}^1)$, a decorated ordered set partition for the figure on the right. The winding number does not change.

**Remark 2.15.** The condition $n \notin T$ is essential for Lemma 2.13. Without this condition, the correspondence changes the winding number as shown in Figure 5.
The winding number on the left figure is 1 but the winding number on the right is 2. We spread elements in blocks in increasing order but since there is a cyclic symmetry, "increasing" might not be meaningful if \( n \in T \).

Figure 5. Correspondence in Lemma 2.13 for \( T = \{1, 7\} \) and \( r = 2 \).

Now fix \( T = \{t_1 < t_2 < \cdots < t_m\} \subseteq \{1, 2, \cdots, n\} \) such that \( n \notin T \). For \( S \in \text{UP}(T) \), the correspondence in Lemma 2.13 gives an embedding

\[
i_S : K_r(S) \hookrightarrow K_r(\{\{t_1\}, \{t_2\}, \cdots, \{t_m\}\})
\]

Let \( \chi_S : K_r(\{\{t_1\}, \{t_2\}, \cdots, \{t_m\}\}) \rightarrow \{0, 1\} \) to be the characteristic function of \( i_S(K_r(S)) \). In other words, \( \chi_S(P) = 0 \) if \( P \notin i_S(K_r(S)) \) and \( \chi_S(P) = 1 \) if \( P \in i_S(K_r(S)) \). Then

\[
H_r(T) = \sum_{S \in \text{UP}(T)} (-1)^{|S|} |K_r(S)| = \sum_{S \in \text{UP}(T)} (-1)^{|S|} |i_S(K_r(S))|
\]

\[
= \sum_{S \in \text{UP}(T)} (-1)^{|S|} \left( \sum_{P \in K_r(\{\{t_1\}, \{t_2\}, \cdots, \{t_m\}\})} \chi_S(P) \right)
\]

(2.5) \[
= \sum_{P \in K_r(\{\{t_1\}, \{t_2\}, \cdots, \{t_m\}\})} \left( \sum_{S \in \text{UP}(T)} (-1)^{|S|} \chi_S(P) \right). \]

Proposition 2.16. If $P$ does not have increasing $r$-packed sequence of length greater than 1, then \[ \sum_{S \in \mathcal{UP}(T)} -(1)^{|S|} \chi_S(P) = (-1)^{|T|}. \] Otherwise it is zero.

Proof. For $P \in K_r(\{\{t_1\}, \{t_2\}, \ldots, \{t_m\}\})$, define $\hat{S}(P)$ to be unordered partition of $T$ by putting $t_i$ and $t_j$ in same part if they belong to same increasing $r$-packed sequence (this will partition $T$ by maximal increasing $r$-packed sequences of $P$). If $\chi_S(P) = 1$, then $S$ should be finer partition than $\hat{S}(P)$. When $P$ has no increasing $r$-packed sequence of length greater than 1, $\hat{S}(P) = \{\{t_1\}, \{t_2\}, \ldots, \{t_m\}\}$, the finest unordered partition of $T$. So $\chi_S(P) = 1$ only when $S = \hat{S}(P)$ thus \[ \sum_{S \in \mathcal{UP}(T)} -(1)^{|S|} \chi_S(P) = (-1)^{|T|}. \] Now assume there is $M \in \hat{S}(P)$ such that $|M| = a \geq 2$. To split $M$ into $b$ parts such that resulting finer partition $S$ still satisfies $\chi_S(P) = 1$, we should choose $(b - 1)$ spots among $(a - 1)$ spaces between adjacent elements of $M$ and put bars in those spots to split $M$. So there are total $\binom{a - 1}{b - 1}$ ways to do that. Then we have

\[
\sum_{S \in \mathcal{UP}(T)} -(1)^{|S|} \chi_S(P) = \prod_{M \in \hat{S}(P), |M| \geq 2} \left( \sum_{b=1}^{|M|} (-1)^b \binom{|M| - 1}{b - 1} \right) \prod_{M \in \hat{S}(P), |M| = 1} (-1).
\]

Since $\sum_{b=1}^{|M|} (-1)^b \binom{|M| - 1}{b - 1} = 0$, \[ \sum_{S \in \mathcal{UP}(T)} -(1)^{|S|} \chi_S(P) = 0 \] whenever $P$ has increasing $r$-packed sequence of length greater than 1, that is, $\hat{S}(P)$ has part with more than one element.

Example 2.17. For $T = \{1, 2, 3, 4\}$, assume $P \in K_r(\{\{1\}, \{2\}, \{3\}, \{4\}\})$ has (maximal) increasing $r$-packed sequence $\{\{1\}, \{2\}, \{3\}, \{4\}\}$. We will list $S \in \mathcal{UP}(T)$ such that $\chi_S(P) = 1$ by number of elements.

$|S| = 1 \rightarrow \{\{1\}, \{2\}, \{3\}, \{4\}\}$

$|S| = 2 \rightarrow \{\{1\}, \{2, 3\}, \{3, 4\}\}, \{\{1, 2\}, \{3\}, \{4\}\}$

$|S| = 3 \rightarrow \{\{1\}, \{2\}, \{3, 4\}\}, \{\{1, 2\}, \{3\}, \{4\}\}$

$|S| = 4 \rightarrow \{\{1\}, \{2\}, \{3\}, \{4\}\}$

So we have \[ \sum_{S \in \mathcal{UP}(T)} -(1)^{|S|} \chi_S(P) = -1 + 3 - 3 + 1 = -3 + 3 - 3 + 3 = 0. \]

Let $\hat{K}_r(T)$ be a subset of $K_r(\{\{t_1\}, \{t_2\}, \ldots, \{t_m\}\})$ consisting of decorated ordered set partition without increasing $r$-packed sequence of length greater than 1. By Proposition 2.16 and (2.5), we have $H_r(T) = (-1)^{|T|} |\hat{K}_r(T)|$. We will count the number of elements in $\hat{K}_r(T)$ by defining the second winding vector for each element. The second winding vector is a modified version of winding vector that we previously defined.

Assume we are given $P \in K_r(\{\{t_1\}, \{t_2\}, \ldots, \{t_m\}\})$. There are $k$ spots total on the circle including empty spots that are recording distances and $T$-singlet blocks $\{t_1\}, \{t_2\}, \ldots, \{t_m\}$ are $r$-bad blocks so for each $\{t_i\}$, there will be at least $(r - 1)$ empty spots after $\{t_i\}$ as the distance to the next block is at least $r$. Color these $r$ spots, that is, the spot occupied by $\{t_i\}$ with $(r - 1)$ empty spots after that red. Doing this for all $i$, $r|T| = rm$ spots will be colored red. And color the remaining $(k - rm)$ spots blue.

Define second winding vector $v = (v_1, v_2, \ldots, v_n)$, by setting $v_i$ to be the number of blue spots passed while moving from $i$ to $(i + 1)$ in clockwise fashion. Do not
include the starting point but include the arriving point (if it’s blue) and when the starting point and the arriving point are in same block (spot), set \( v_i = 0 \). Since the winding number is \( d \), the whole path winds around the circle \( d \) times. So we have \( v_1 + \cdots + v_n = (k - rm)d \).

If \( i \notin T \), we are starting from the blue spot so \( v_i \) can range from 0 to \( (k - rm - 1) \). However when \( i \in T \), we claim \( v_i \) cannot be zero. If \( v_i = 0 \), then path from \( i \) to \( i + 1 \) should not include any blue spots. So the path will be of the form \( \{i\}, \phi, \cdots, \phi, \{a_1\}, \phi, \cdots, \phi, \{i+1\} \) where \( \phi \) means an empty spot. Thus the sequence \( (\{i\}, \{a_1\}, \cdots, \{a_q\}, \{i+1\}) \) is \( r \)-packed. Since \( P \) does not have increasing \( r \)-packed sequence of length greater than 1, \( (i, a_1, \cdots, a_2, i + 1) \) should be a decreasing sequence which is impossible. It is possible to have \( v_i = k - rm \) since \( i \) is not in the blue spot. We conclude \( 1 \leq v_i \leq k - rm \).

**Example 2.18.** Figure 6 shows how to read off the second winding vector. We are given \( T = \{1, 2, 9\} \), and \( r = 2 \). The upper left figure is a picture for the decorated ordered set partition \( (\{2\}_2, \{1\}_2, \{5, 6\}_1, \{7, 8\}_1, \{9\}_3, \{11, 12, 13\}_1, \{10, 14\}_1, \{3, 4\}_1) \). Note that the sequence \( (\{2\}, \{1\}) \) is \( r \)-packed but not increasing \( r \)-packed. So \( P \) has no increasing \( r \)-packed sequence of length greater than 1. After coloring spots with the rule above we get the upper right figure. There will be \( r|T| = 6 \) red spots and \( k - r|T| = 6 \) blue spots. To get \( v_1 \), wind from 1 to 2 clockwise as shown in the lower figure, and count the number of blue spots passed. Here \( v_1 = 6 \). Continuing this process we have the second winding vector \( v = (6, 6, 0, 1, 0, 1, 0, 3, 5, 0, 0, 1, 1) \).

![Figure 6. Reading off the second winding vector.](image)

We saw that a second winding vector \( v = (v_1, v_2, \cdots, v_n) \) satisfies \( v_1 + \cdots + v_n = (k - rm)d \), \( 0 \leq v_i \leq k - rm - 1 \) if \( i \notin T \), and \( 1 \leq v_i \leq k - rm \) if \( i \in T \).
It turns out these are the only restrictions for the second winding vectors of the elements of $\hat{K}_r(T)$.

**Proposition 2.19.** Elements of $\hat{K}_r(T)$ are in bijection with elements of
\[
\{(v_1, v_2, \cdots, v_n) \in \mathbb{Z}^n \mid 0 \leq v_i \leq k - rm - 1 \text{ if } i \notin T, 1 \leq v_i \leq k - rm \text{ if } i \in T, v_1 + \cdots + v_n = (k - rm)d\}.
\]

**Proof.** The forward direction is done by the second winding vector. For the reverse direction, we should recover the decorated ordered set partition (in $\hat{K}_r(T)$) whose second winding vector is the specified vector. First draw $(k - rm)$ spots on the circle (recall $|T| = m$) and put 1 in one spot. Having put $i$ in some spot, move clockwise $w_i$ spots and put $i + 1$ in that spot. After placing every element, let’s denote the resulting decorated ordered set partition with $P$. We construct $\tilde{P} \in \hat{K}_r(T)$ as follows.

For each block $B$ of $P$ with $B \cap T \neq \emptyset$, let $B \cap T = \{i_1 < \cdots < i_s\}$. We replace $B$ with $B \setminus T$ and then add $rs$ spots immediately after $B \setminus T$ as follows: first a $T$-singlet block $\{i_s\}$ then $(r - 1)$ empty spots then $T$-singlet block $\{i_{s-1}\}$ then $(r - 1)$ empty spots $\cdots$ $T$-singlet block $\{i_1\}$ then $(r - 1)$ empty spots. \Halmos

**Example 2.20.** Figure 7 shows how to recover a decorated ordered set partition from a second winding vector as stated in Proposition 2.19. We are given $T = \{1, 2, 9\}$, $r = 2$, and the second winding vector $v = (6, 6, 0, 1, 0, 1, 0, 3, 5, 0, 0, 1, 1)$. In the upper left figure, there are $6 = k - r|T|$ spots ($k = 12$) on the circle and 1 is in one spot. Then put elements according to the second winding vector. The upper right figure shows this. The elements in $T$ are denoted with a tilde. Consider the first block of elements in $T$.

![Figure 7](image-url)
block \{1,2,3,4\}. The numbers 3 and 4 will form a block and 1 and 2 will spread to the right into the space between blocks \{1,2,3,4\} and \{5,6\}, making four new red spots. The same thing happens for the block \{7,8,9\}, making two new red spots. The lower figure is the picture for the resulting decorated ordered set partition in \(\hat{K}_r(T)\). We recovered Example 2.18.

For a second winding vector \(v = (v_1, \cdots, v_n)\), let \(v' = (v'_1, \cdots, v'_n)\) be a vector such that \(v'_i = v_i\) if \(i \notin T\), and \(v'_i = v_i-1\) if \(i \in T\). By the property of a second winding vector, we have \(0 \leq v'_i \leq k-rm-1\) and \(v'_1 + \cdots + v'_n = (k-rm)d - |T| = (k-rm)d - m\). So the number of such \(v'\) is \(\binom{n}{(k-rm)d-m} k-rm\) which gives

\[
H(T) = (-1)^{|T|} |\hat{K}_r(T)| = (-1)^m \binom{n}{(k-rm)d-m} k-rm.
\]

Proof of Conjecture 1.5) By Proposition 2.10 and (2.6), the number of \(r\)-hypersimplicial decorated ordered set partitions (of type \((k,n)\) with winding number \(d\)) is

\[
\sum_{T \subseteq \{1,2,\ldots,n\}} H_r(T) = \sum_{m \geq 0} (-1)^m \binom{n}{m} \binom{n}{(k-rm)d-m} k-rm.
\]

Now compare with the formula (2.4). □

Acknowledgements: The author would like to thank Lauren Williams for pointing out this problem and her helpful comments on drafts of this paper, and Melissa Sherman-Bennet for helping me revise this paper. The author is also grateful to Nick Early for helpful explanations about the background of this conjecture.

References


E-mail address: donghyun_kim@berkeley.edu
E-mail address: dhkim@math.harvard.edu